Toward Affine Recognition of Handwritten Mathematical Characters

Oleg Golubitsky[(1)], Vadim Mazalov[(2)] and Stephen M. Watt[(2)]

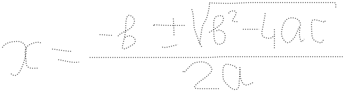
(1) Google, Inc.
340 Hagey Blvd., Waterloo, ON, Canada.

(2) Ontario Research Center for Computer Algebra
Department of Computer Science
University of Western Ontario, London, ON, Canada

International Workshop on Document Analysis Systems
June 9-11, 2010
Boston, MA, USA

Introduction

- We are working towards online recognition of handwritten mathematics in pen-based environment.



$\longrightarrow$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- In the previous work, we achieved 96.3% recognition rate of handwritten mathematical characters, subjected to rotation.

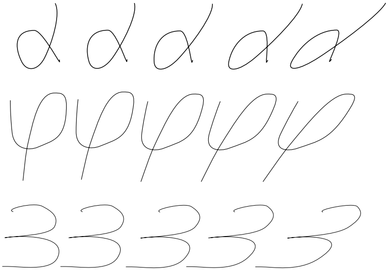
- Now the question raised, whether the algorithm can include shear-invariance without drop in the classification rate. We answer this question positively.

Organization of the Presentation

- Shear.
- Integral invariants.
- Size normalization of samples.
- Parameterization of coordinate functions.
- Classification Algorithm.
- Experimental Setting.
- Approximation of Invariants.
- Classification results.
- Mixed parameterization.
- Towards affine recognition.
- Conclusion.

Shear

- Commonly occurs in practice, possibly even more often than rotation.



- Easily recognizable by a human, even for a large degree of transformation (more than 1 radian).
- Therefore, in practice, a large amount of shear should be expected.

Shear: difficulties

- Parameterization by arc length is not suitable, since the length of a curve changes under shear.
- Size normalization is not trivial.
- Shear may easily transform symbols to different characters.

Integral Invariants

- Provide an elegant approach to planar and spatial curves classification under affine transformations.
- Relatively insensitive to local random noise, as opposed to differential invariants.
- Computed from coordinate functions, which are approximated online, i.e. when the curve is being written, with a minor overhead after pen-up.

As the name suggests, integral invariants are constructed from quantities obtained by integration.

Functional Representation

- I_1 and I_2 are invariant under special linear group $SL(2, R)$ and defined as:

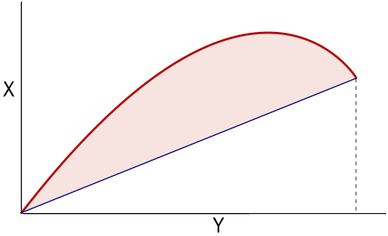
$$I_1(\lambda) = \int_0^\lambda X(\tau)dY(\tau) - \frac{1}{2}X(\lambda)Y(\lambda)$$

$$I_2(\lambda) = X(\lambda)\int_0^\lambda X(\tau)Y(\tau)dY(\tau) - \frac{1}{2}Y(\lambda)\int_0^\lambda X^2(\tau)dY(\tau) - \frac{1}{6}X^2(\lambda)Y^2(\lambda)$$

where $X(\lambda)$, $Y(\lambda)$ are coordinate functions.

Geometric Representation

- Invariant $I_1(\lambda)$ is the area between the curve and a secant.



- Invariant $I_2(\lambda)$ can be described in terms of volume.

Numerical Representation

- After approximation of coordinate functions with truncated Legendre-Sobolev (L-S) polynomial series, invariants take this form

$$I_1(\lambda) \approx \sum_{i,j=1}^{d} x_i y_j \left[\int_0^\lambda P_i(\tau) P_j'(\tau) d\tau - \frac{1}{2} P_i(\lambda) P_j(\lambda) \right]$$

$$I_2(\lambda) \approx \sum_{i,j,k,l=1}^{d} x_i x_j y_k y_l \mu_{ijkl}$$

$$\mu_{ijkl} = P_i(\lambda) \int_0^\lambda P_j(\tau) P_k(\tau) P_l'(\tau) d\tau$$
$$- \frac{1}{2} P_l(\lambda) \int_0^\lambda P_i(\tau) P_j(\tau) P_k'(\tau) d\tau$$
$$- \frac{1}{6} P_i(\lambda) P_j(\lambda) P_k(\lambda) P_l(\lambda).$$

Coefficients of Invariants

- Taking the discussed representation of $I_1(\lambda)$, coefficients of the invariant are computed as

$$I_{1,i} = \frac{\langle I_1, P_i \rangle}{\langle P_i, P_i \rangle} \quad i = 1..d$$

where $\langle \cdot, \cdot \rangle$ is the L-S inner product.

- Similarly, we calculate coefficients for $I_2(\lambda)$ and obtain a $2d$-dimensional vector for each sample

$$(I_{1,1}, \ldots, I_{1,d}, I_{2,1}, \ldots, I_{2,d}).$$

Size Normalization: Existing Methods that we test

- Taking the Euclidean norm of the vector of L-S coefficients of coordinate functions.
- Normalizing by the height of a sample (works for horizontal shear, but not for rotation).
- Aspect-ratio size normalization (may work for rotation, but becomes inaccurate for horizontal shear).

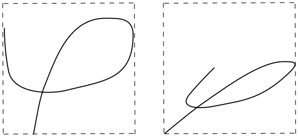


Figure: Aspect ratio size normalization.

Size Normalization: Our Approach

- For the case of shear (and affine) recognition we look at the norm $\|I_1\|$ of the coefficient vector of I_1.

- Coefficients of coordinate functions are normalized by multiplication by $1/\sqrt{\|I_1\|}$.

- Computing the norm of I_1 allows to extend the invariance of I_1 and I_2 from the special linear group, $SL(2, R)$, to the general linear group, $GL(2, R)$. Invariance under the general affine group, $Aff(2, R)$, is obtained by dropping the first (order-0) coefficients from the coefficient vectors of the coordinate functions.

Linear Symbols

Taking the norm of I_1 may perform poorly for samples that have a linear shape, such as "-", "\", "/", "l", ".". Being an area between the curve and its secant, I_1 is close to zero for such characters.

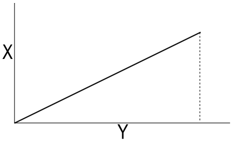


Figure: I_1 for a linear symbol.

Therefore, some of the methods, designed for 2-dimensional curves, are indeed not suitable for 1-dimensional symbols, and such linear characters require special treatment.

Parameterization of Coordinate Functions

We test the following parameterization approaches

- By time.
- By Euclidean arc length

$$F(\lambda) = \int_0^{\lambda} \sqrt{(X'(\tau))^2 + (Y'(\tau))^2} d\tau.$$

- By affine arc length

$$\hat{F}(\lambda) = \int_0^{\lambda} \sqrt[3]{|X'(\tau)Y''(\tau) - X''(\tau)Y'(\tau)|} d\tau.$$

Classification: Forming Convex Hulls

- Classification is based on the distance to convex hulls of nearest neighbours in the space of L-S coefficients.
- To form convex hulls, nearest neighbours are selected with Manhattan distance

$$d = \sum_{i=1}^{n} |x_i - y_i|$$

- Distance to the convex hulls is evaluated with the square Euclidean distance.

Classification: Selecting the Class

- We select N classes in the space of L-S coefficients of the integral invariants.

- We consider each of the selected classes C_i to find the minimal distance to the subject sample with respect to different levels of shear in the space of L-S coefficients of the coordinate functions:

$$\min_{\phi} \mathrm{CHNN}_k(X(\phi), C_i),$$

where $X(\phi)$ is the sheared image of the test sample curve X and $\mathrm{CHNN}_k(X, C)$ is the distance from a point X (in the L-S space) to the convex hull of k nearest neighbors in class C.

Experimental Setting

- Our database contains 50,703 samples, collected at ORCCA (special mathematical characters, Latin letters and digits), UNIPEN and LaViola datasets.
- The number of strokes is included in the class labels (thus, single-stroke and double-stroke "7" are considered as two different classes).
- Therefore, the total number of classes is raised from 242 to 378.
- To increase precision, integral invariants were partially approximated in Maple, using rational arithmetic.

Approximation of Invariants

Maximum absolute and average relative errors in coefficients of invariants of initial samples and sheared by 1 radian

	I_1		I_2	
Degree	Abs. Err.	Rel. Err.	Abs. Err.	Rel. Err.
2	9×10^{-12}	3×10^{-19}	3×10^{-11}	9×10^{-20}
3	1×10^{-11}	4×10^{-19}	8×10^{-10}	2×10^{-19}
4	5×10^{-11}	9×10^{-19}	1×10^{-9}	4×10^{-19}
5	6×10^{-11}	3×10^{-18}	3×10^{-9}	1×10^{-18}
6	3×10^{-10}	1×10^{-17}	9×10^{-9}	5×10^{-18}
7	2×10^{-9}	5×10^{-17}	7×10^{-8}	2×10^{-17}
8	3×10^{-8}	2×10^{-16}	1×10^{-7}	1×10^{-16}
9	2×10^{-7}	1×10^{-15}	6×10^{-7}	5×10^{-16}
10	2×10^{-6}	6×10^{-15}	4×10^{-6}	2×10^{-15}
11	5×10^{-6}	3×10^{-14}	2×10^{-5}	1×10^{-14}
12	1×10^{-5}	1×10^{-13}	7×10^{-5}	6×10^{-14}
13	4×10^{-5}	7×10^{-13}	5×10^{-4}	3×10^{-13}
14	3×10^{-4}	4×10^{-12}	3×10^{-3}	1×10^{-12}
15	1×10^{-3}	2×10^{-11}	7×10^{-3}	8×10^{-12}
16	1×10^{-2}	1×10^{-10}	2×10^{-2}	5×10^{-11}
17	5×10^{-2}	5×10^{-10}	3×10^{-2}	6×10^{-10}
18	4×10^{-1}	3×10^{-9}	3×10^{-1}	4×10^{-9}

We take degree=12.

Classification

Classification results for different types of parameterization of coordinate functions: by time, by Euclidean arc length (AL) and by affine arc length (AAL) for discussed size normalization approaches.

(a) Size normalization by height

	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
AAL	82.2	82.2	82.2	82.1	82.1	82.1	82.1	82.1	82.1	82
AL	96.4	96.4	96.1	95.6	95	94.1	93	91.9	90.2	88
Time	94.8	94.9	94.9	94.7	94.5	94.4	94.4	94.4	94.4	94.3

(b) Aspect ratio size normalization

	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
AAL	81.9	81.8	81.6	81.4	81.2	81	80.8	80.2	79.4	77.4
AL	96.3	96.4	96.1	95.5	94.7	93.7	92.3	90.1	85.7	77.5
Time	94.7	94.7	94.6	94.3	94.1	93.9	93.7	93.2	91.9	89.

(c) Size normalization by l_1

	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
AAL	83	83.1	83	82.9	82.9	82.8	82.8	82.8	82.8	82.7
AL	**96.3**	**96.3**	**96.1**	**95.7**	**95.1**	**94.4**	**93.3**	**91.9**	**90.2**	**87.9**
Time	94.6	94.7	94.6	94.5	94.5	94.5	94.5	94.5	94.5	94.4

Mixed Parameterization

Parameterization by time gives low recognition rate, while remains affine-invariant. It is opposite for parameterization by arc length. We, therefore, propose to unite these two parameterization approaches in the form of mixed parameterization as follows

- Divide the curve in N equal time intervals, and parameterize each interval by arc length.
- Smooth the transition from time to arc length with a mixed metric of the form $kdt^2 + dx^2 + dy^2$ inside the subintervals, where k is a parameter.
- The optimal values of N and k are found by cross-validation.

Recognition rate for different N and k

We increased the number of selected classes to 50 and obtained the following rates

Table: Recognition rate (%) for mixed parameterization for corresponding values of N and k

	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$N = 2,\ k = 1$	96.1	96.2	96	95.9	95.8	95.7	95.6	95.3	95.2	94.9
$N = 2,\ k = 2$	95.8	95.9	96	95.8	95.7	95.7	95.7	95.6	95.5	95.5
$N = 3,\ k = 0.5$	96.1	96.2	96	95.9	95.9	95.7	95.5	95.6	95.3	95
$N = 3,\ k = 1$	95.9	96.2	96	95.8	96	95.8	95.7	95.6	95.5	95.2
$N = 4,\ k = 1$	95.9	96.1	95.9	95.7	95.8	95.6	95.6	95.7	95.6	95.4

We take $N = 2,\ k = 2$ as these values allow to obtain high rate for non-sheared samples and stay invariant when significant distortions take place.

Recognition rate for different N and k

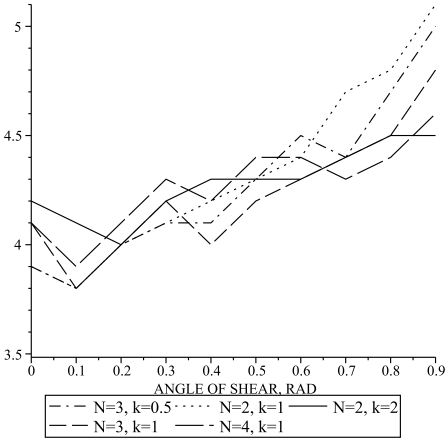


Figure: Error (%) for the mixed parameterization for different values of skew.

Towards Affine Recognition

We propose two approaches to deploy integral invariants for affine recognition

- Consider both rotation and horizontal shear in the same system. The minimization problem would have to include two parameters: angles of rotation and shear.
- Define a distance that consists of weighted coefficients of coordinate functions and integral invariants. Size of a sample may also be included in the distance metrics.

Conclusion

We have developed

- Size normalization of samples, when shear (and more generally, affine) transformations take place.
- Mixed parameterization of coordinate functions that allows to obtain high recognition rate, while being invariant to affine distortions.
- Shear-invariant algorithm that in conjunction with rotation-invariance (introduced in the previous work) allows to model recognition of samples, subjected to the most common affine transformations.

References

1. S. Feng, I. Kogan and H. Krim, Classification of Curves in 2D and 3D via Affine Integral Signatures, to appear in *Acta Appl. Math.*, 2008, `http://arxiv.org/abs/0806.1984`.